RECOGNIZING G-INDUCED FLOWS

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ABSTRACT

A flow on a homogeneous space need not come from the action of a one-parameter subgroup of the ambient transitive group. We exhibit an inductive method for determining when this is the case.

1. Introduction

Let M be a connected, compact C^∞ manifold. By a flow on M we shall mean a C^∞ action of the real numbers on M. Suppose that x is a flow on M, the effect of $t \in \mathbf{R}$ on $m \in M$ being denoted $x_t(m)$. Suppose further that $\mathbf{G}$ is a connected Lie group. We shall say that the flow x is $\mathbf{G}$-induced if we can find a C^∞, transitive action $(g, m) \mapsto gm$ of $\mathbf{G}$ on M with the property that for some one-parameter subgroup ϕ: $\mathbf{R} \to \mathbf{G}$ of $\mathbf{G}$, we have $x_t(m) = \phi(t)m$ for all $t \in \mathbf{R}$ and $m \in M$.

Because $\mathbf{G}$-induced flows are particularly easy to study, it is reasonable to seek a criterion for determining when a given flow is $\mathbf{G}$-induced for some appropriate $\mathbf{G}$. One inductive approach to finding such a criterion runs along the following lines:

Suppose that, in addition to M, we are given a second connected, compact C^∞ manifold N. Let us also suppose that there is a connected Lie group $\mathbf{H}$ that operates smoothly on N so that quotient $N/\mathbf{H}$ is M. A flow on N that permutes the orbits of $\mathbf{H}$ will then project to a flow on M. One is thus led to pose the inductive problem: Let x be a flow on N such that x permutes the orbits of $\mathbf{H}$, and such that the flow on M obtained by projecting x is $\mathbf{G}$-induced. Under what conditions can one then conclude that there is an extension $1 \to \mathbf{H} \to \mathbf{L} \to \mathbf{G} \to 1$ of $\mathbf{G}$ by $\mathbf{H}$ such that x is $\mathbf{L}$-induced?

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In the generality this problem has just been posed, it is hopelessly difficult. If, however, one restricts one's attention to the case in which $\mathbf{G}$ and $\mathbf{H}$ are abelian, then one can get remarkably complete results, as we shall show. Our condition involves the solvability of a certain differential equation associated with the projected flow on M. When the equation is solvable, $\mathbf{L}$ can be found. In Section 3, we show that when the equation is not solvable, $\mathbf{L}$ need not exist. We leave the precise formulation of the results to the next section.

2. The main result

Let us begin by establishing some notation. As above, N will denote a connected compact C^∞ manifold. We shall assume given an action of $\mathbf{R}^r$ on N, and to keep the notation as in the Introduction, we shall use $\mathbf{H}$ to denote $\mathbf{R}^r$ in this connection. In order to get positive results, one must impose rather stringent conditions on the action of $\mathbf{H}$ on N:

We assume that there is a tower $\{0\} = \mathbf{H}_0 \subseteq \mathbf{H}_1 \subseteq \cdots \mathbf{H}_r = \mathbf{H}$ *of connected subgroups of* $\mathbf{H}$ *with the following properties*:

(1) The dimension of each $\mathbf{H}_k$ is k, so that $\mathbf{H}_k/\mathbf{H}_{k-1} = \mathbf{R}$. (2) The action of $\mathbf{H}$ on N induces an action of $\mathbf{H}_k/\mathbf{H}_{k-1}$ on $N/\mathbf{H}_{k-1}$. We assume that the subgroup of $\mathbf{H}_k/\mathbf{H}_{k-1}$ leaving a point in $N/\mathbf{H}_{k-1}$ fixed is isomorphic to the integers $\mathbf{Z}$ *and does not depend on the point.* (3) Set $M_k = N/\mathbf{H}_k$. Using standard arguments from [4], one can see that the condition (2) on the action of $\mathbf{H}$ implies the existence of a C^∞ manifold structure on M_k such that the natural map $p_k \colon N \to M_k$ is C^∞. In particular, $M_r = N/\mathbf{H}_r = N/\mathbf{H}$ is a C^∞ manifold. Our final assumption is that M_r is diffeomorphic to the n-dimensional torus $\mathbf{T}^n$.

In keeping, again, with the notation of the Introduction, we shall usually write M in place of M_r. Thus, $M = N/\mathbf{H}$.

Let $x \colon \mathbf{R} \times N \to N$ be a flow. By the derivative of x, we shall mean, as is usual, the vector field ∂_x on N whose effect on $f \in C^\infty(N)$ is given by:

$$\partial_x f(n) = \lim_{t\to 0}[f(x_t(n)) - f(n)]/t.$$

Since $\mathbf{H}$ acts on N, each one-parameter subgroup $\phi \colon \mathbf{R} \to \mathbf{H}$ of $\mathbf{H}$ defines a flow on N, namely: $(t, n) \mapsto \phi(t)n$. We shall indulge in a slight abuse of notation and use ∂_ϕ to denote the derivative of this flow. We shall use H to denote the real vector space spanned by the vector fields ∂_ϕ, where ϕ varies over all one-parameter subgroups of $\mathbf{H}$. Similarly, H_k will denote the real vector space spanned by the ∂_ϕ's with ϕ a one-parameter subgroup of $\mathbf{H}_k$.

We shall use $T(N)$ to denote the real vector space of global C^∞ vector fields on N. (Similar notation will be used for other manifolds as well.) Given X and Y from $T(N)$, we denote by $[X, Y]$ their Lie bracket. We shall use $H_k^\#$ to denote $\{X \in T(N): [X, H_m] \subseteq H_{m-1} \text{ for } m = 1, 2, \cdots k\}$. If $f \in C^\infty(M_k)$ and $X \in H_k^\#$, then the function $X(f \circ p_k)$ on N is constant on orbits of $\mathbf{H}_k$ and hence defines a function $(p_k^* X)f$ in $C^\infty(M_k)$. We thus get a linear map $p_k^*: H_k^\# \to T(M_k)$.

LEMMA 2.1. *Let $\{X_1, \cdots, X_r\}$ be a basis for H over $\mathbf{R}$ such that for each k, $\{X_1, \cdots, X_k\}$ is a basis for H_k. The only elements of $H_k^\#$ annihilated by p_k are, then, those of the form $\alpha_1 X_1 + \cdots + \alpha_k X_k$, where each α_j is a C^∞ function on N constant on orbits of $\mathbf{H}_k$.*

Furthermore, p_k^* maps $H_k^\#$ onto $T(M_k)$.

PROOF. Our conditions on the action of $\mathbf{H}$ guarantee that at each point $m \in N$ the vectors $X_1(m), \cdots, X_r(m)$ are linearly independent over $\mathbf{R}$. From this, it follows by an easy induction that the kernel of p_k^* is as asserted. What requires some effort is proving that p_k^* is surjective.

Let $q_k: M_{k-1} \to M_k$ denote the natural map. Conditions (1) and (2) on the action of $\mathbf{H}$ imply that the action of $\mathbf{H}_k/\mathbf{H}_{k-1}$ on M_{k-1} induces an action of $\mathbf{T}$ (the circle group) on M_{k-1} such that for each $m \in M_k$, the group $\mathbf{T}$ acts simply transitively on $q_k^{-1}(m)$. It follows that each point in M_k lies in an open set U such that $q_k^{-1}(U)$ is diffeomorphic with $U \times \mathbf{T}$ via a diffeomorphism that intertwines the given action of $\mathbf{T}$ on $q_k^{-1}(U)$ with the natural action $\mathbf{T}$ on $U \times \mathbf{T}$ ("Ehresmann's lemma", see [2: 31].) Now, using a partition of unity argument, one can easily show that for each $X \in T(M_k)$ there exists some $Y \in T(M_{k-1})$ such that $[Y, p_{k-1}^* X_k] = 0$ and such that Y projects via q_k onto X. The proof of the lemma is completed by induction on k. Q.E.D.

Our main result is the following theorem.

THEOREM. 2.2. *Let x be a flow on N whose derivative lies in $H_r^\# = H^\#$, and let y be the flow on M obtained by projecting x. If*

(i) *the flow y has at least one dense orbit in M,*

(ii) *the flow y on M is $\mathbf{R}^n$-induced, and*

(iii) *for each $g \in C^\infty(M)$ there exists a constant b and a function $f \in C^\infty(M)$ such that $g = \partial_y f + b$, then there is an extension $1 \to \mathbf{H} \to \mathbf{L} \to \mathbf{R}^n \to 1$, with $\mathbf{L}$ nilpotent, such that the flow x on N is $\mathbf{L}$-induced.*

PROOF. Each one-parameter subgroup $\psi: \mathbf{R} \to \mathbf{R}^n$ yields a flow $(t, m) \mapsto \psi(t) \cdot m$ on M, the derivative of which will be denoted ∂_ψ. The real vector space spanned by the various vector fields ∂_ψ will be denoted G. Set $B_1 = \partial_y$, which is in G by assumption, and choose $B_2, \cdots, B_n$ from G so that $\{B_1, \cdots, B_n\}$ is a basis for G over $\mathbf{R}$. Also, set $Y_1 = \partial_x$. Our objective is to find elements $Y_2, \cdots, Y_n$ in $H^{\#}$ such that for each $j \leqq n$, we have $p^*Y_j = B_j$, and further, such that the real vector space L spanned by $Y_1, \cdots, Y_n$ and H is a Lie subalgebra of $T(N)$. Once we have L, it will follow from our construction and the definition of $H^{\#}$ that L is nilpotent, and it will follow from a result of Palais [4: 73] that the unique connected, simply connected Lie group $\mathbf{L}$ whose Lie algebra is L acts as desired on N.

From Lemma 2.1 we know at least that there are elements $Z_2, \cdots, Z_n$ in $H^{\#}$ satisfying $p^*Z_j = B_j$. Our problem is to "perturb" the Z_j's so that we get vector fields that bracket with one another (and with Y_1) properly.

The Jacobi identity yields that each $[Y_1, Z_j]$ is in $H^{\#}$. Hence we may form $p^*[Y_1, Z_j]$. But $p^*[Y_1, Z_j] = [p^*Y_1, p^*Z_j] = [B_1, B_j] = 0$, and therefore we may apply Lemma 2.1 to conclude that for each j, the bracket $[Y_1, Z_j]$ is $\Sigma_{k=1}^{r} g_{jk} X_k$, where $\{X_1, \cdots, X_r\}$ is the basis for H described in Lemma 2.1, and each g_{jk} is a C^∞ function on N constant on orbits of $\mathbf{H}$.

We now use condition (iii) on $B_1 = \partial_y$. That condition implies that for each pair (j,k) with $2 \leqq j \leqq n$ and $1 \leqq k \leqq r$, there exist constants b_{jk} and functions $f_{jk} \in C^\infty(M)$ such that, when g_{jk} is viewed as a function on M, we have

$$B_1 f_{jr} = b_{jr} - g_{jr} \quad \text{and} \qquad (2.2.1)$$

$$B_1 f_{jk} = b_{jk} - g_{jk} - \sum_{m=k+1}^{r} c_{km} f_{jm}$$

for $1 \leqq k \leqq r-1$, where $c_{2m}, c_{3m}, \cdots c_{rm}$ are constants defined by $[Y_1, X_m] = \Sigma_{k=1}^{m-1} c_{km} X_k$. Note that in finding b_{jk} and f_{jk}, we must proceed inductively from $k = r$ to $k = r-1$, etc.

Set $f_{jk}^* = f_{jk} \circ p$, and set $Y_j = Z_j + \Sigma_{k=1}^{r} f_{jk}^* X_k$. It then follows from (2.2.1) that for $2 \leqq j \leqq n$,

$$[Y_1, Y_j] = \sum_{k=1}^{r} b_{jk} X_k. \qquad (2.2.2)$$

Of course, since each b_{jk} is in $\mathbf{R}$, we have $[Y_1, Y_j] \in H$. Also, since $X f_{jk}^* = 0$ for all $X \in H$, and since $[H, H] = 0$, we have that for $2 \leqq j \leqq n$,

$$[Y_j, X] = [Z_j, X] \in H \qquad (2.2.3)$$

for all $X \in H$. That $[Y_1, H] \subseteq H$ is part of the hypothesis of the theorem, since $Y_1 = \partial_x$.

Let L denote the real vector space spanned by $Y_1, \cdots, Y_n$ and H. We want to show that L is a Lie subalgebra of $T(N)$, that is, we want $[L, L] \subseteq L$. In view of (2.2.2) and (2.2.3), it remains to prove that $[Y_i, Y_j] \in L$ when $2 \leqq i, j \leqq n$. We will actually show that $[Y_i, Y_j] \in H$. (We need this to guarantee that L is nilpotent.)

It follows from (2.2.3) and from $Z_j \in H$ that $Y_j \in H$. Hence $[Y_i, Y_j] \in H$, and we may form $p^*[Y_i, Y_j]$, which is readily seen to be 0. Using Lemma 2.1, we see that there are functions $h_k \in C^\infty(N)$ constant on orbits of $\mathbf{H}$ such that $[Y_i, Y_j] = \sum_{k=1}^r h_k X_k$. Thus,

$$[Y_1, [Y_i, Y_j]] = \sum_{k=1}^{r} \{(Y_1 h_k) X_k + h_k [Y_1, X_k]\}. \tag{2.2.4}$$

On the other hand, Jacobi's identity together with (2.2.2) and (2.2.3) yield that $[Y_1, [Y_i, Y_j]] \in H$, which means that there are constants $d_1, \cdots, d_r$ such that

$$[Y_1, [Y_i, Y_j]] = \sum_{k=1}^{r} d_k X_k. \tag{2.2.5}$$

At this point we finally need condition (i), that y has a dense orbit in M. This condition implies that if $f \in C^\infty(M)$ and if $\partial_y f = 0$, then f is constant. Since M is compact, we can strengthen condition (i) to say that if $f \in C^\infty(M)$ and if $\partial_y f$ is constant, then f is constant. In the presence of this condition on ∂_y, the right-hand sides of (2.2.4) and (2.2.5) can be equal *only if each h_k is constant*, whence $[Y_i, Y_j] = \sum_{k=1}^r h_k X_k \in H$.

Hence $[L, L] \subseteq H \subseteq L$. Q.E.D.

The power of Theorem 2.1 lies in the ready verifiability of condition (iii). We shall have more to say about that momentarily. Notice that condition (ii) is not something we are required to verify: it is the "induction hypothesis" on which we are trying to build. As for condition (i), it is easily seen to be a consequence of (ii) and (iii)—our discussion of condition (iii) will show this clearly.

Recall that M can be realized as the n-dimensional torus $\mathbf{T}^n = \mathbf{R}^n/\mathbf{Z}^n$ in such a way that for some $v \in \mathbf{R}^n$, the flow y is given by $y_t(w + \mathbf{Z}^n) = (w + tv) + \mathbf{Z}^n$. (This is condition (ii).) It follows that ∂_y is just the directional derivative ∂_v. Hence an analysis of condition (iii) comes down to looking at ordinary directional derivatives. Now condition (iii) asserts that ∂_v be as nearly surjective as possible. Thus, verifying condition (iii) amounts to inverting the operator ∂_v, and the standard approach to such problems is to use Fourier analysis. Let f be a function on $\mathbf{R}^n/\mathbf{Z}^n$ with Fourier series expansion $f(w) \sim \Sigma_\mu a_\mu \exp 2\pi i\langle \mu, w\rangle$, the sum being over $\mu \in \mathbf{Z}^n$, and $\langle \mu, w\rangle$ denoting the usual inner product of $w \in \mathbf{R}^n$ with μ. Then $f \in C^\infty(\mathbf{R}^n/\mathbf{Z}^n)$ if, and only if, for every positive integer k, we have that

$$\sup_\mu (1 + \langle \mu, \mu\rangle)^k |a_\mu| \tag{2.3.1}$$

is finite. Suppose now that $f, g \in C^\infty(\mathbf{R}^n/\mathbf{Z}^n)$ and $b \in \mathbf{R}$. Let $\Sigma_\mu b_\mu \exp 2\pi i\langle \mu, w\rangle$ be the Fourier series of g. In order for $\partial_v f + b = g$ to hold, we must have

$$\{2\pi i \Sigma_\mu a_\mu \langle \mu, v\rangle \exp 2\pi i\langle \mu, w\rangle\} + b = \Sigma_\mu b_\mu \exp 2\pi i\langle \mu, w\rangle,$$

which is equivalent to

$$\begin{cases} 2\pi i\langle \mu, v\rangle a_\mu = b_\mu \text{ for } \mu \neq 0 \text{ in } \mathbf{Z}^n \\ b = b_0. \end{cases} \tag{2.3.2}$$

Hence, condition (iii), as a condition on v, says that whenever $\{b_\mu : \mu \neq 0 \in \mathbf{Z}^n\}$ are the Fourier coefficients of a C^∞ function, then $\{b_\mu/(2\pi i\langle \mu, v\rangle) : \mu \neq 0 \in \mathbf{Z}^n\}$ are also the Fourier coefficients of a C^∞ function, in particular, the condition requires $\langle \mu, v\rangle \neq 0$ whenever $\mu \in \mathbf{Z}^n$ and $\mu \neq 0$. The only problem is that $\langle \mu, v\rangle$ might be non-zero but very small (compared to $\langle \mu, \mu\rangle^{-k}$), so that division by $\langle \mu, v\rangle$ destroys the rapid decay needed (as in (2.3.1)) to get a C^∞ function. Note that this is no problem when $n = 1$, so condition (iii) is always satisfied when $n = 1$. For $n \geqq 2$, the situation is more complicated:

We shall say that $v \in \mathbf{R}^n$ ($n \geqq 2$) is *poorly approximable* if there exists a positive integer k such that

$$\inf_{\mu \neq 0} |\mu|^k |\langle \mu, v\rangle|$$

is strictly positive; otherwise we shall say that v is well approximable.

THEOREM 2.3. *Let $v \in \mathbf{R}^n$. Then v is poorly approximable if, and only if, for every $g \in C^\infty(\mathbf{R}^n/\mathbf{Z}^n)$ one can find a constant b and a function $f \in C^\infty(\mathbf{R}^n/\mathbf{Z}^n)$ satisfying $\partial_v f + b = g$.*

REMARK. This result is surely well known.

PROOF OF THEOREM 2.3. It follows by comparing (2.3.1) and (2.3.2) that when v is poorly approximable, the equation $\partial_v f + b = g$ can always be solved for f and b. Suppose, conversely, that v is well approximable. It may then happen that $\langle \mu, v\rangle = 0$ for some non-zero μ, and in that case, if we take $g(w) = \exp 2\pi i\langle \mu, w\rangle$, we cannot solve $\partial_v f + b = g$. Hence the only troublesome case is when v is well approximable, but $\langle \mu, v\rangle \neq 0$ for all non-zero $\mu \in \mathbf{Z}^n$:

Since v is well approximable, we can find for each positive integer k a non-zero $\mu_k \in \mathbf{Z}^n$ satisfying

$$\langle \mu_k, \mu_k\rangle^k \left|\langle \mu_k, v\rangle\right| < k^{-1}. \tag{2.3.3}$$

Define g to be the function on $\mathbf{R}^k/\mathbf{Z}^k$ whose μth Fourier coefficient b_μ is given by

$$\begin{cases} b_{\mu_k} = \langle \mu_k, v\rangle \text{ for } k = 1, 2, \cdots \\ b_\mu = 0 \text{ if } \mu \text{ is not one of the } \mu_k\text{'s.} \end{cases} \tag{2.3.4}$$

Combining (2.3.1) and (2.3.3), we see that $g \in C^\infty(\mathbf{R}^n/\mathbf{Z}^n)$. Also, $g \neq 0$, because $\langle \mu, v\rangle \neq 0$ when $\mu \neq 0$. It is, on the other hand, clear from (2.3.1) and (2.3.2) that $\partial_v f + b = g$ has no solution with $f \in C^\infty(\mathbf{R}^n/\mathbf{Z}^n)$. Q.E.D.

It is not difficult to see that, apart from a set of measure zero in $\mathbf{R}^n$, every $v \in \mathbf{R}^n$, $n \geqq 2$, satisfies $\inf_\mu \langle \mu, \mu\rangle^n \left|\langle \mu, v\rangle\right| > 0$. Thus almost all of $\mathbf{R}^n$ is poorly approximable. (See [1, §I.3].) There are, however, a host of well approximable vectors. Consider the situation in $\mathbf{R}^2$:

Let W denote the set of all well approximable $v \in \mathbf{R}^2$ that also satisfy $\langle \mu, v\rangle \neq 0$ for all non-zero $\mu \in \mathbf{Z}^2$. Whether $v = (v_1, v_2)$ is in W depends only on how well the ratio v_1/v_2 is approximable by rational numbers. For example, if one sets $v(1) = 10$ and defines $v(k)$, $k \geqq 2$, inductively by $v(k) = v(k-1)!$, and if τ denotes $\sum_{k=1}^{\infty} 10^{-v(j)}$, then $(1, \tau) \in W$.

Now that we know W is not empty, it becomes necessary to check whether Theorem 2.2 remains true if we drop condition (iii). It does not. The next section gives a counterexample.

3. An example

Let $\mathbf{N}$ denote the Lie group whose underlying manifold is $\mathbf{R}^3$ and whose group operation is given in terms of the coordinates in $\mathbf{R}^3$ by $(x, y, z)(t, u, v) = (x + t, y + u, z + v + xu)$. The subset Γ of $\mathbf{N}$ consisting of all points with *integer* coordinates is actually a subgroup of $\mathbf{N}$, and $\mathbf{N}/\Gamma$ is a compact manifold that will play the role of N. We shall take $\mathbf{H} = \mathbf{R}$, the action $t \in \mathbf{H}$ on $(x, y, z)\Gamma \in N$ being given by $t \cdot (x, y, z)\Gamma = (x, y, z + t)\Gamma$. The quotient $M = N/\mathbf{H}$ is $\mathbf{T}^2$. Note that since $\dim(H) = 1$, we have $H^\# = \{X \in T(N) : [X, H] = 0\}$. Let Z be a non-zero element of H.

LEMMA 3.1. *Let $X \in H^{\#}$ satisfy*: (1) $Xf = 0$ *only if f is constant, and* (2) *the projection of X to a vector-field on M is ∂_v for some $v \in \mathbf{R}^2$. If U and V belong to $T(N)$, if*

$$[X, U] = V \text{ and } [X, V] = [U, V] = 0,$$

and if for each $x \in N$ the vectors $X(x)$, $U(x)$, and $V(x)$ are linearly independent, then V lies in H and U lies in $H^{\#}$.

PROOF. There exist three functions χ, η, and $\zeta \in C^{\infty}(N)$ such that $Z = \chi X + \eta V + \zeta U$. It follows that $[X, Z] = (X\chi)X + (X\eta + \zeta)V + (X\zeta)U$. On the other hand, $X \in H^{\#}$ and thus $[X, Z] = 0$, whence $X\chi = X\zeta = \zeta + X\eta = 0$. Hence χ and ζ are constants, and $X\eta = -\zeta$ implies that $\zeta = 0$ and η is constant. We have thus proved that Z is in the real linear span of X and V. Since $[V, X] = 0$, it follows that $[V, Z] = 0$—in other words, $V \in H^{\#}$. Applying Lemma 2.1, we see that, since $V \in H^{\#}$, we can prove $V \in H$ simply by proving that $p^*V = 0$.

We have already seen that $Z = \chi X + \eta V$ for some constants χ and η. Furthermore η must be non-zero, by virtue of condition (1) on X. Now $p^*Z = 0$ and $p^*X = \partial_v$, which implies that $p^*V = -(\chi/\eta)\partial_v$. Thus what we need to prove is $\chi = 0$.

Condition (1) on X implies that if $f \in C^{\infty}(M)$ and $\partial_v f = 0$, then f is constant. Hence the flow on M whose derivative is ∂_v is the Kronecker flow, with all orbits dense. Let $q: \mathbf{R} \times N \to N$ denote the flow whose derivative is V. We are going to show that the orbits of q are compact. Plainly this is consistent with $p^*V = -(\chi/\eta)\partial_v$ only if $\chi = 0$. Thus, proving q has compact orbits will prove the lemma.

It is a result of Palais [4, p. 73] that there is an action of $\mathbf{N}$ on N whose derivative is the real Lie algebra spanned by $\{X, U, V\}$. This action is clearly transitive and thus is merely another presentation of N as $\mathbf{N}/\Gamma$. The center of $\mathbf{N}$ acts on N with derivative V, which tells us that q is just the flow defined by the center of $\mathbf{N}$. This flow obviously has compact orbits. Q.E.D.

LEMMA 3.2. *Let X, U, and V remain as in Lemma 3.1. Then $p^*U = \partial_w$ for some $w \in \mathbf{R}^2$.*

PROOF. Recall that $p^*X = \partial_v$. Choose $w \in \mathbf{R}^2$ so that $\{v, w\}$ is a basis for $\mathbf{R}^2$. Then there are functions α and β in $C^{\infty}(M)$ such that $p^*U = \alpha\partial_v + \beta\partial_w$. Note that if we can show that $\partial_v\alpha = \partial_v\beta = 0$, then we will have shown that α and β are constant (condition (1) on X, again), which proves the lemma.

We have just seen that $p^*V = 0$. Since $V = [X, U]$, it follows that

$$0 = [p^*X, p^*U] = [\partial_v, \alpha\partial_v + \beta\partial_w] = (\partial_v\alpha)\partial_v + (\partial_v\beta)\partial_w.$$

Hence $\partial_v\alpha = \partial_v\beta = 0$, as desired. Q.E.D.

We are now going to build a flow on N that is not $\mathbf{N}$-induced, but that projects to an $\mathbf{R}^2$-induced flow on M. It is not hard to see that $\mathbf{N}$ is the only connected, simply connected nilpotent group of dimension 3 that operates transitively on N. Thus our flow, which will satisfy conditions (i) and (ii) of Theorem 2.2 by construction, is not $\mathbf{L}$-induced for any nilpotent $\mathbf{L}$ that sits in an extension of the form $1 \to \mathbf{H} \to \mathbf{L} \to \mathbf{R}^2 \to 1$. (Note that $\mathbf{H}$ is one-dimensional in our present situation.)

Let $v \in \mathbf{R}^2$ be well approximable, and assume also that $\langle \mu, v \rangle \neq 0$ for all non-zero $\mu \in \mathbf{Z}^2$. Choose a poorly approximable $w = (w_1, w_2) \in \mathbf{R}^2$ so that $v_1 w_2 - v_2 w_1 = 1$, where $v = (v_1, v_2)$. By Theorem 2.3, we can find $g \in C^\infty(\mathbf{T}^2)$ such that $\partial_v f + b = g$ has no solution with $f \in C^\infty(\mathbf{T}^2)$ and b constant. On the other hand, we *can* find $f \in C^\infty(\mathbf{T}^2)$ and a constant b so that $\partial_w f + b = g$. We will use this bad g, good f, and good b to build the flow.

Recall that $N = \mathbf{N}/\Gamma$. Let X^* and Y^* denote, respectively, the derivatives of the flows

$$(t, (x, y, z)\Gamma) \mapsto (x + t, y, z + ty)\Gamma$$

and

$$(t, (x, y, z)\Gamma) \mapsto (x, y + t, z)\Gamma.$$

Then $[X^*, Y^*]$ is a non-zero element of H, which, without harm, we may assume is Z.

Define $X \in T(N)$ by $X = v_1 X^* + v_2 Y^* + (f \circ p)Z$, where p, as always, is the natural projection $N \to M$. The vector-field X is the derivative of a flow q on N. By a standard argument (see [3: 54, statement 2.6]), the flow q has a dense orbit. Note also that $[X, Z] = 0$, and thus $X \in H^\#$.

THEOREM 3.3. *The flow q is not* $\mathbf{N}$*-induced.*

PROOF. Suppose that the theorem is false. We can then find U and V in $T(N)$ so that $\{X, U, V\}$ satisfies the hypotheses of Lemma 3.1. Replacing U (if need be) by a real linear combination of X and U, we can, by virtue of Lemma 3.2, arrange that $p^*U = \partial_w$, where w is the element of $\mathbf{R}^2$ used to define X. The vector field U will then have the form $w_1 X^* + w_2 Y^* + hZ$ for some C^∞ function h on N that is constant on $\mathbf{H}$ orbits. We also have, by Lemma 3.1, that $V = cZ$ for some non-zero constant c. Hence

$$cZ = [X, U] = (1 + \partial_v h - \partial_w f)Z.$$

Since $\partial_w f + b = g$, it follows that

$$\partial_v h + (b - c + 1) = g. \tag{3.3.1}$$

However, (3.3.1) stands in flat contradiction to our assumption that $\partial_v h$ + *constant* $= g$ has no C^∞ solution h. Q.E.D.

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